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MATHEMATICAL STUDY OF NON-UNIFORM HEAT SOURCE ON MHD HEAT TRANSFER IN A LIQUID FILM OVER AN UNSTEADY STRETCHING SHEET

Ananthaswamy. V.
Department of Mathematics
The Madura College (Autonomous)
Madurai, Tamil Nadu, India

Yogeswari. N.
M. Phil., Mathematics
The Madura College (Autonomous)
Madurai, Tamil Nadu, India

Abstract— **In this paper, an analysis of heat source on MHD flow and heat transfer to a laminar liquid film form an impervious stretching sheet. The basic running equations are in the form of partial differential equations. Using similarity transformation, these equations are converted to non-linear ordinary differential equations. The approximated analytical solutions of the dimensionless velocity and dimensionless temperature are derived by using the New Homotopy analysis method. The graphical representations of axial flow and temperature profiles are depicted with the help of magnetic parameter, unsteadiness parameter, dimensionless film thickness and prandtl number. This method can be easily extended to solve other non-linear and boundary value problems in other MHD flows.**

Keywords— Unsteady stretching surface; Similarity transformation; Magnetic parameter; New Homotopy analysis method.

I. INTRODUCTION

The analysis of flow a thin liquid film had interested to study of number of researches because of its suitable applications in science and technology. Other applications such as aerodynamic extrusion of plastic sheets and fibers, drawing, annealing and thinning of copper wire and glass blowing. The problem of extrusion of thin surface layers needs attention to income some idea for controlling the coating product efficiently. We studied the analytical boundary layer flow over a stretching sheet by Crane [1]. We have referred to the three dimensional flow due to a stretching surface and that due to a stretching surface in a rotating fluid by Wang [2]. We analyzed the heat transfer in a viscoelastic boundary layer flow over a stretching sheet with viscous dissipation and non-uniform heat source by Abel et al [3].

In study of Wang [4] to the case of finite fluid domain are extended by several authors by Usha et al [5]-[8] for fluids of both Newtonian and non-Newtonian kinds using various and thermal boundary conditions. There are extensive works in literature concerning the production of thin fluid film either on a vertical wall achieved through the action of gravity or that over a rotating disc achieved through the action of centrifugal forces. The thin liquid film flow over a rotating horizontal disk by Dandapat and Ray[9].Easily, the unsteady MHD film over a rotating disk by Kumari and Nath[10].The viscoelastic fluid flow and heat transfer over a stretching sheet under the effects of a non-uniform heat source, viscous dissipation and thermal radiation by Cortell[11]

Most of the aforementioned studies have neglected the combined effect of non-uniform heat source/sink and magnetic field on the heat transfer which is important in view point of desired properties of the outcome. In the present study we include the same for heat transfer analysis in a thin liquid film from an unsteady stretching sheet.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

In this section we consider a thin elastic sheet which emerges from a narrow slit at the origin of a Cartesian co-ordinate system for investigations in Fig.1.The continuous sheet at $y = 0$ is parallel with the x-axis and moves in its own plane with the velocity as follows

$$U(x,t) = \frac{bx}{(1-\alpha t)} \tag{1}$$

Here b and α are both positive constants with dimensions per time. The surface temperature T_s of the stretching sheet is assumed to vary with the distance x from the slit as

$$T_s(x,t) = T_0 - T_{ref}\left[\frac{bx^2}{2\nu}\right](1-\alpha t)^{-3/2} \tag{2}$$

Here T_0 denotes as the temperature at the slit and T_{ref} refers to a constant reference temperature such that $0 \le T_{ref} \le T_0$. The equation $bx^2/\nu(1-\alpha t)$ can be recognized as the local Reynolds number based on the surface velocity U. From eqn.(1) for the velocity of the sheet $U(x,t)$ reflects that the elastic sheet which is fixed at the origin is stretched by applying a force in the positive x-direction and the effective stretching rate $b/(1-\alpha t)$ increase with time as $0 \le \alpha \le 1$.

The analysis with the same expression for the surface temperature $T_s(x,t)$ given eqn.(2) shows a situations in which the sheet temperature decreases from T_0 at the slit in proportion to x^2 and such that the amount of temperature reduction along the sheet increases with time. The applied transverse magnetic field is assumed to be of variable kind and is chosen in its special form as below

$$B(x,t) = B_0(1-\alpha t)^{-1/2} \tag{3}$$

The particular form of the expressions for $U(x,t), T_s(x,t)$ and $B(x,t)$ are chosen so as to facilitate the construction of a new similarity transformation which enables in transforming the governing partial differential equations of momentum and heat transport into a set of non-linear ordinary differential equations.

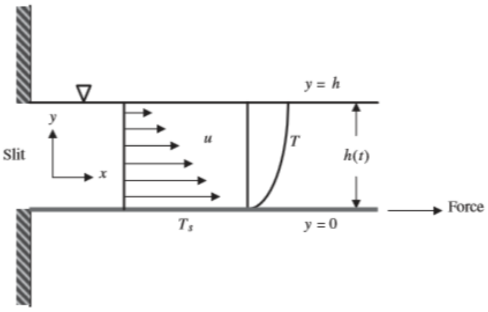

Fig.1: Physical representation of a liquid film on an unsteady stretching sheet.

Let us consider a thin elastic liquid film of uniform thickness $h(t)$ lying on the horizontal stretching sheet. The *x*-axis is chosen in the direction along which the sheet is set to motion and the y-axis is taken perpendicular to it. The fluid motion within the film is primarily caused solely by stretching of the sheet. The sheet is stretched by the action of two equal and opposite forces along *x*-axis. The sheet is assumed to have velocity U as defined in eqn.(1) and the flow field is exposed to the influence of an external transverse magnetic field of strength B as defined in eqn.(3). We have neglected the effect of latent heat due to evaporation by assuming the liquid to be non-volatile. Further the buoyancy is neglected due to the relatively thin liquid film, but it is not so thin that intermolecular forces come into play. The velocity and temperature fields of the liquid film obey the following boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u \tag{5}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{q'''}{\rho C_p} \tag{6}$$

The non -uniform heat source/sink is modeled as

$$q''' = \frac{k u_w(x)}{x\nu}\left[A^*(T_s - T_0)f' + (T - T_0)B^*\right], \tag{7}$$

Here A^* and B^* are the coefficients of space and temperature dependent heat source/sink respectively. In this case $A^* > 0, B^* > 0$ corresponds to internal heat generation and that $A^* < 0, B^* < 0$ corresponds to internal heat absorption.

The pressure in the surrounding gas phase is assumed to be uniform and the gravity force gives rise to a hydrostatic pressure variation in the liquid film. In order to justify the boundary layer approximation, the length scale in the primary flow direction must be significantly larger than the length scale in the cross stream direction. We choose the representative measure of the film thickness to be $(\nu/b)^{1/2}$ so that the scale ratio is large enough. That is $x/(\nu/b)^{1/2} >> 1$. This choice of length scale enables us to employ the boundary layer approximations. Further it is assumed that the induced magnetic field is negligibly small. The boundary conditions are given below

$$u = U, \quad v = 0, \quad T = T_s \quad at \quad y = 0 \tag{8}$$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 \qquad at \quad y = h \tag{9}$$

$$v = \frac{dh}{dt} \qquad at \quad y = h \tag{10}$$

At this juncture we make a note that the mathematical problem is implicitly formulated only for $x \ge 0$. Further it is assumed that the surface of the planar liquid film is smooth so as to avoid the complications due to surface waves. The influence of interfacial shear due to the quiescent atmosphere, in other words the effect of surface tension, is assumed to be negligible. The viscous shear stress $\tau = u(\partial u/\partial y)$ and the heat flux $q = -k(\partial T/\partial y)$ vanish at the adiabatic free surface (at $y = h$).

Consider dimensionless variable f and θ, the similarity variable η as

$$f(\eta) = \frac{\psi(x,y,t)}{\left(\frac{\nu b}{1-\alpha t}\right)^{1/2} x} \tag{11}$$

$$\theta(\eta)=\frac{T_0-T(x,y,t)}{T_{ref}(\frac{bx^2}{2v(1-\alpha t)^{-3/2}})} \tag{12}$$

$$\eta=\left(\frac{b}{v(1-\alpha t)}\right)^{1/2} y \tag{13}$$

The physical stream function $\psi(x,y,t)$ automatically assures mass conversion given in eqn.(4). The velocity components as follows

$$u=\frac{\partial\psi}{\partial y}=\left(\frac{bx}{1-\alpha t}\right)f'(\eta) \tag{14}$$

$$v=-\frac{\partial\psi}{\partial x}=-\left(\frac{vb}{1-\alpha t}\right)^{1/2} f(\eta) \tag{15}$$

The mathematical problem defined in eqn. (4)-(6),(8) and (9) transforms exactly into a set of differential equations and their boundary conditions:

$$S(f'+\frac{\eta}{2}f'')+(f')^2-ff'=f'''-Mnf' \tag{16}$$

$$pr\left[\frac{S}{2}(3\theta+\eta\theta')+2\theta f'-\theta' f\right]=\theta''+(A^* f'+B^*\theta) \tag{17}$$

$$f'(0)=1, \quad f(0)=0, \quad \theta(0)=1 \tag{18}$$

$$f''(\beta)=0 \quad \theta'(\beta)=0 \tag{19}$$

$$f(\beta)=\frac{S\beta}{2} \tag{20}$$

Where $S\equiv\alpha/b$, the dimensionless measure of the unsteadiness and the prime is indicates differentiation with respect to η. Further β represents the value of the similarity variable η at the free surface so that eqn.(13) below

$$\beta=\left(\frac{b}{v(1-\alpha t)}\right)^{1/2} h \tag{21}$$

But β is an unknown constant, which should be determined as an integral part of the boundary value problem. The rate at which film thickness varies can be obtained differentiating eqn.(21)
With respect to t, in the form

$$\frac{dh}{dt}=-\frac{\alpha\beta}{2}(\frac{v}{b(1-\alpha t)})^{1/2} \tag{22}$$

Therefore the kinematic constraint at $y=h(t)$ given by eqn. (10) transforms into the free surface condition (22). It is noteworthy that the momentum boundary layer defined by eqn.(16) subject to the relevant boundary conditions (18)-(20) is decoupled from the thermal field; on the other hand the temperature field $\theta(\eta)$ is coupled with the velocity field $f(\eta)$. Since the sheet is stretched horizontally the convection least affects the flow and hence there is a one-way coupling of velocity and thermal fields.
The local skin friction coefficient, which is of practical importance, is as follows

$$C_f\equiv\frac{-2\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho U^2}=-2Re^{-1/2}{}_x f''(0) \tag{23}$$

and the heat transfer between the surface and the fluid convention ally expressed in dimensionless form as a local Nusselt number is given by

$$Nu_x\equiv-\frac{x}{T_{ref}}\left(\frac{\partial T}{\partial y}\right)_{y=0}=\frac{1}{2}(1-\alpha t)^{-1/2}\theta'(0)Re_x^{3/2} \tag{24}$$

Here $Re_x=Ux/v$ is the local Reynolds number and T_{ref} represents the same reference temperature as in eqn.(2). We find the solution of the boundary value problem (16)-(20).

III. SOLUTION OF THE PROBLEMS USING THE NEW HOMOTOPY ANALYSIS METHOD

Homotopy analysis method (HAM) is a non-perturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to numerous problems in science and engineering [11-33]. In comparison with other perturbative and non-perturbative analytical methods, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, HAM has proved to be the most effective method for obtaining analytical solutions to highly non-linear differential equations. Previous applications of HAM have mainly focused on non-linear differential equations in which the non-linearity is a polynomial in terms of the unknown function and its derivatives. As seen in (1), the non-linearity present in electro hydrodynamic flow takes the form of a rational function, and thus, poses a greater challenge with respect to finding approximate solutions analytically. Our results show that even in this case, HAM yields excellent results.
Liao [12-20] proposed a powerful analytical method for non-linear problems, namely the Homotopy analysis method. This method provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. In order to investigate the accuracy of the Homotopy analysis method (HAM) solution with a finite number of terms, the system of differential equations were solved. The Homotopy analysis method is a good technique comparing to another perturbation method. The Homotopy analysis method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. Using this method, we can obtain the following solution to (1) and (2) (see Appendix B).
The approximate analytical solution of the equations (1) and (2) using HAM is given by

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$$f(\eta)=\eta+\frac{S\beta}{2}-\beta -h\left[\left(\frac{Mn+S+1}{6}\right)\eta^3-\left(\frac{Mn+S+1}{2}\right)\beta\eta^2+\left(\frac{Mn+S+1}{3}\right)\beta^3\right] \quad (25)$$

$$\theta(\eta)=1-h\left[\begin{array}{l}\left(\dfrac{-A^*-B^*+\frac{3prS}{2}+2pr}{2}\right)\eta^2 \\ -\left(\dfrac{A^*(Mn+S+1)}{24}\right)\eta^4 \\ +\left(\dfrac{A^*(Mn+S+1)\beta}{6}\right)\eta^3 \\ +\left(\dfrac{2pr(Mn+S+1)}{24}\right)\eta^4 \\ -\left(\dfrac{2pr(Mn+S+1)\beta}{6}\right)\eta^3 \\ +\left(A^*+B^*-\dfrac{3prS}{2}-2pr\right)\beta\eta \\ -\dfrac{A^*(Mn+S+1)\ \eta\beta^3}{3} \\ +\left(\dfrac{2pr(Mn+S+1)\beta^3}{3}\right)\eta\end{array}\right] \quad (26)$$

IV. RESULTS AND DISCUSSION

In this section the effects of physical parameters including magnetic parameter Mn, unsteadiness parameters S, prandtl number pr, coefficient of space A^*, coefficient of temperature B^*, dimensionless film thickness β will be presented. Fig.1 show that the physical model of a liquid film on an elastic sheet. Fig.2 and 3 illustrates the differential axial flow $(f^{\prime}(\eta))$ versus dimensionless similarity variable (η). From Fig.2, we infer that the dimensionless axial velocity decreases, when the magnetic parameter increases in some fixed values of other dimensionless parameter S,β.From Fig.3, represents the dimensionless axial velocity, when the magnetic parameter increases for some fixed values of other dimensionless parameter S,β. Fig. 4 and 5 illustrates the dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η).From Fig.4, shows that the dimensionless temperature increases, when magnetic parameter increases in some fixed value of other dimensionless parameters S,β,A^*,B^*,pr. From Fig.5, we evident that when the magnetic parameter increases, the corresponding dimensionless temperature decreases in some fixed value of other dimensionless parameters S,β,A^*,B^*,pr.

Fig. 6 and 7 shows the dimensionless temperature $(\theta(\eta))$ with respect to the dimensionless similarity variable (η). From Fig.6, it is noted that when the value of coefficient of space increases, the dimensionless temperature decreases in some fixed value of other parameters S,β,Mn,B^*,pr. From Fig. 7, we depicts that the dimensionless temperature increases, when the coefficient of space increases for some fixed value of other dimensionless parameters S,β,Mn,B^*,pr.

Fig. 8 and 9 describes that the dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). From Fig.8, we demonstrate that when the coefficient of temperature increases, the corresponding dimensionless temperature increases in some fixed value of other dimensionless parameters S,β,A^*,Mn,pr. From Fig.9, we observed that the dimensionless temperature decreases when the coefficient of temperature increases in some fixed value of parameter S,β,A^*,Mn,pr.

From Fig. 10, represents the dimensionless stress parameter $-(f^{\prime\prime}(0))$ w.r.to the magnetic parameter (Mn). This Fig. clearly demonstrates that the increase in the value of magnetic parameter produces decrease in the values of the dimensionless stress parameter for some fixed value of parameters S,β. From Fig. 11, shows that the dimensionless temperature gradient $-(\theta^{\prime}(0))$ versus the magnetic parameter (Mn). It observer that increases in the values of dimensionless temperature gradient, when the value of magnetic parameter decreases in some fixed value of other parameters S,β,A^*,B^*,pr.

From Fig. 12, we infer that the dimensionless temperature gradient $-(\theta^{\prime}(0))$ and coefficient of space (A^*). It describes that dimensionless temperature gradient decrease when coefficient of space increases in some fixed value of parameters From Fig. 13, we noted that the dimensionless temperature gradient $-(\theta^{\prime}(0))$ and coefficient of temperature (B^*). It represents that when the coefficient of temperature increases, the corresponding values of dimensionless temperature gradient decreases for some fixed parameters.

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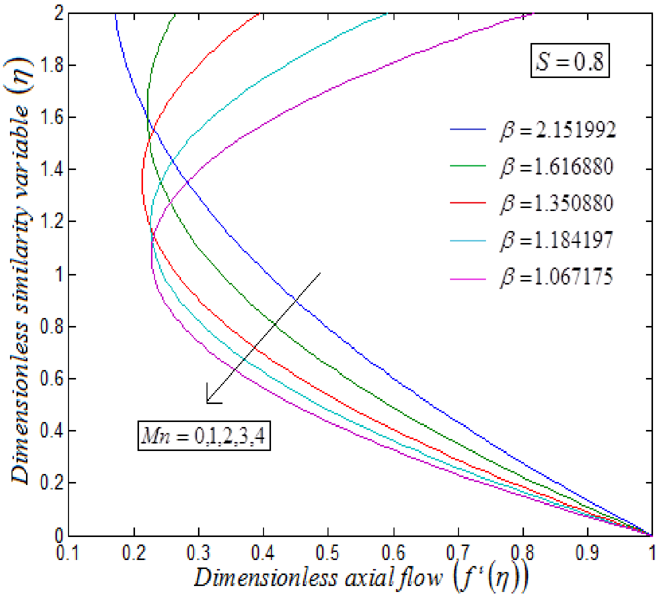

Fig.2: Dimensionless axial flow $(f'(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the dimensionless film thickness (β), magnetic parameter (Mn) and some fixed value of the other parameter S using the eqn.(25) when $h=-0.2209$.

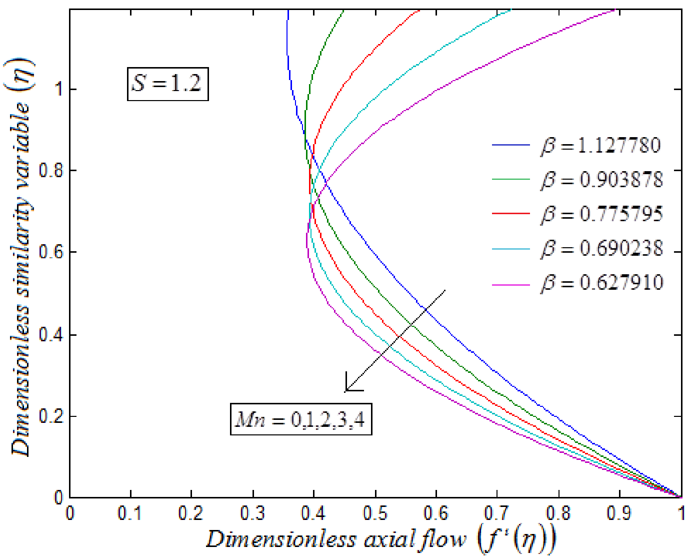

Fig.3: Dimensionless axial flow $(f'(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the dimensionless film thickness (β), magnetic parameter (Mn) and some fixed value of the other parameter S using the eqn.(25) when $h=-0.23$.

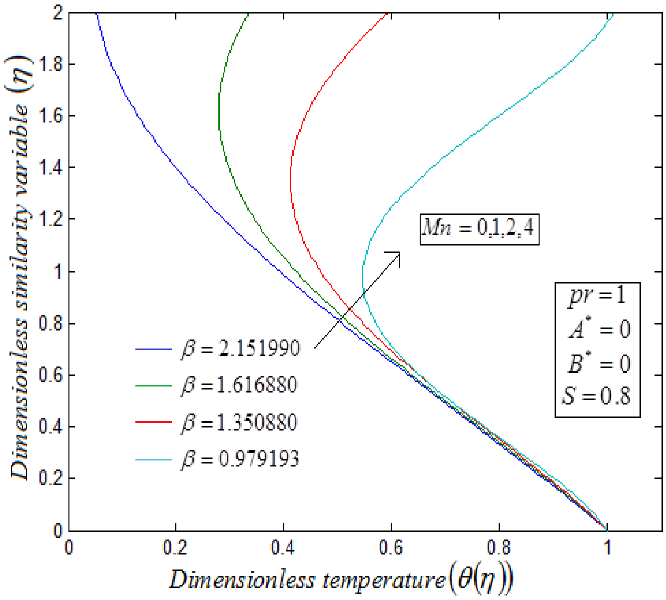

Fig.4: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the dimensionless film thickness (β), magnetic parameter (Mn) and some fixed value of the other parameter S, pr, A^*, B^* using the eqn.(26) when $h=0.3025$.

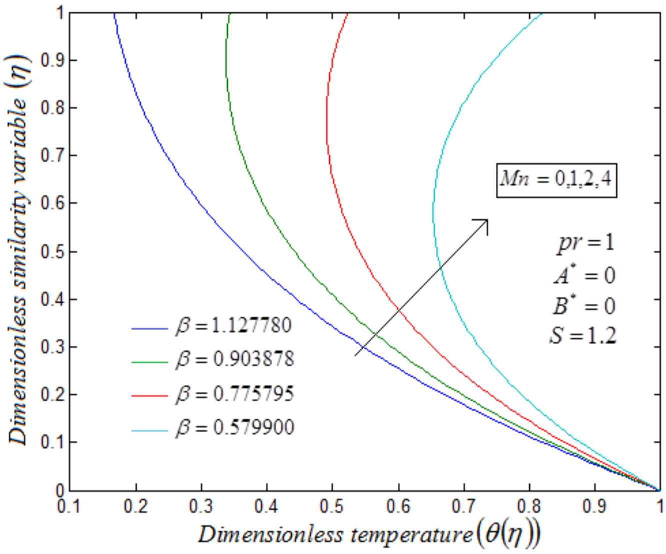

Fig.5: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the dimensionless film thickness (β), magnetic parameter (Mn) and some fixed value of the other parameter S, pr, A^*, B^* using the eqn.(26) when $h=0.3025$.

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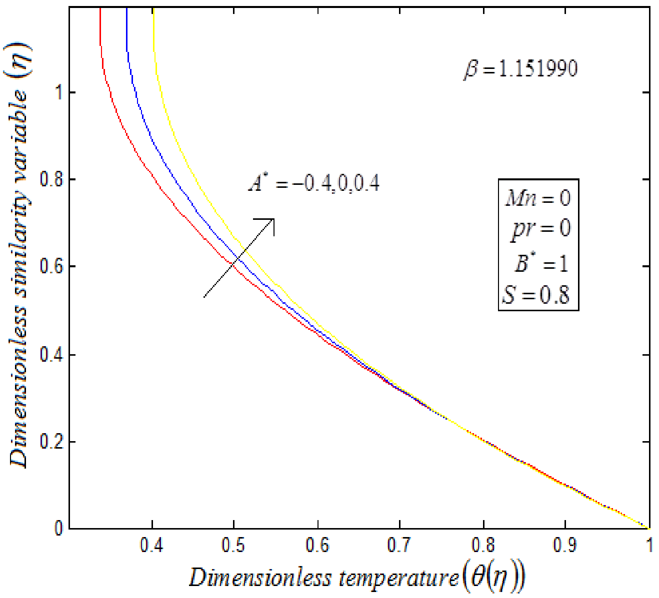

Fig.6: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the coefficient of space (A^*) and some fixed value of the other parameter S, pr, B^*, Mn, β using the eqn.(26) when $h = 0.95$.

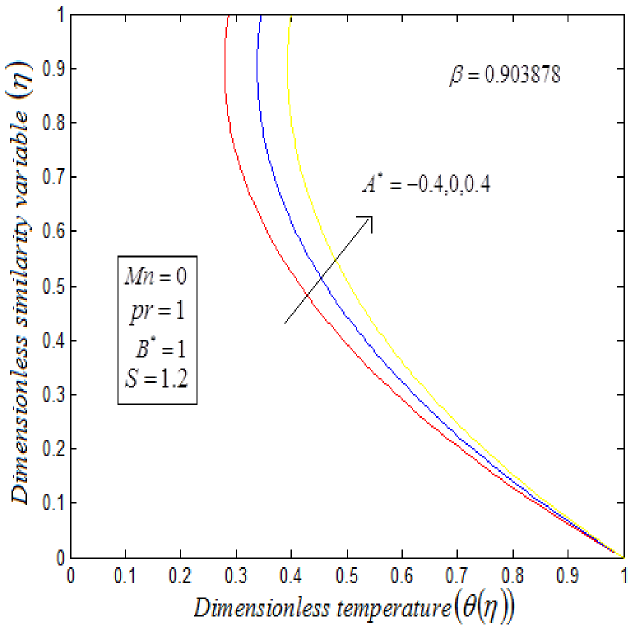

Fig.7: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the coefficient of space (A^*) and some fixed value of the other parameter S, pr, B^*, Mn, β using the eqn.(26) when $h = -0.95$.

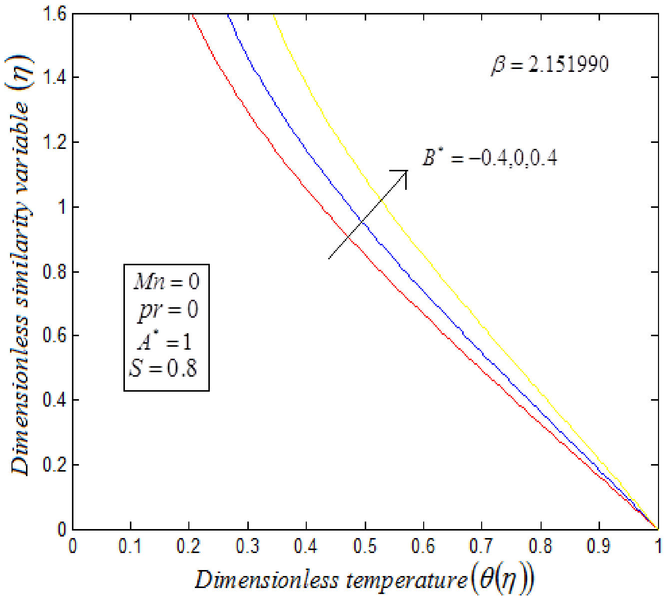

Fig.8: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the coefficient of temperature (B^*) and some fixed value of the other parameter S, pr, A^*, Mn, β using the eqn.(26) when $h = -0.14$.

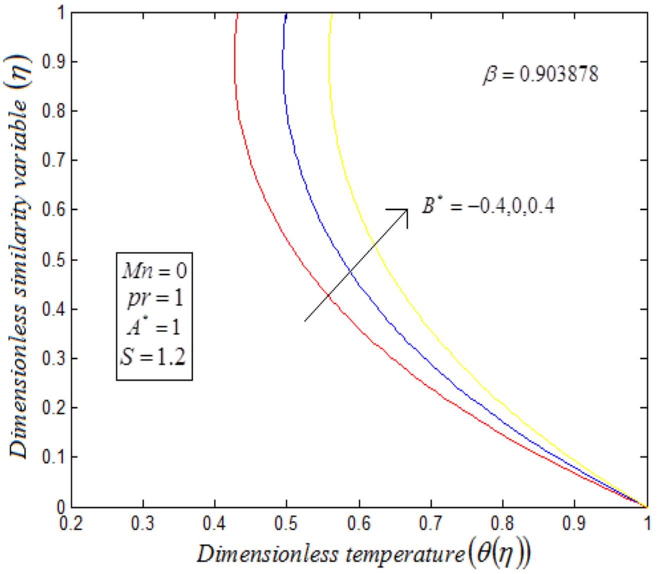

Fig.9: Dimensionless temperature $(\theta(\eta))$ versus dimensionless similarity variable (η). The curves are plotted for various values of the coefficient of temperature (B^*) and some fixed value of the other parameter S, pr, A^*, Mn, β using the eqn.(26) when $h = -0.8033$.

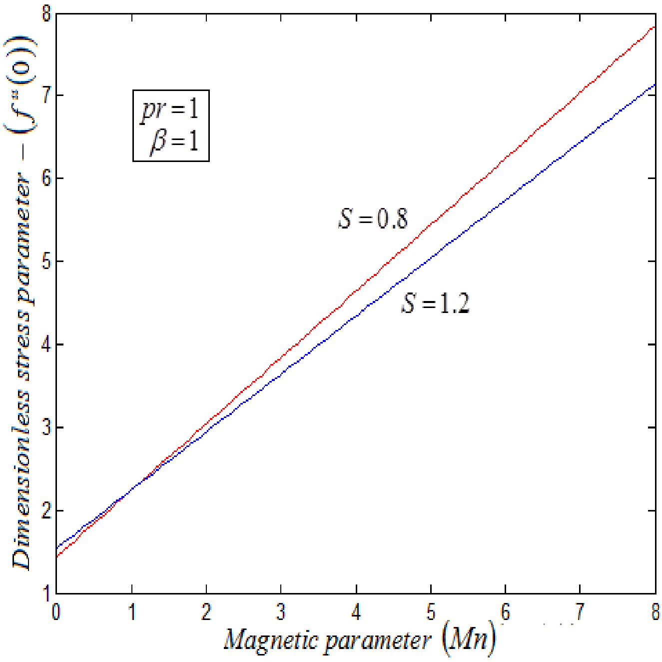

Fig.10: Dimensionless stress parameter $-(f''(0))$ versus magnetic parameter (Mn). The curves are plotted for various values of the coefficient of space (S) and some fixed value of the other parameter pr,β using the eqn.(25) when $h=-0.75$.

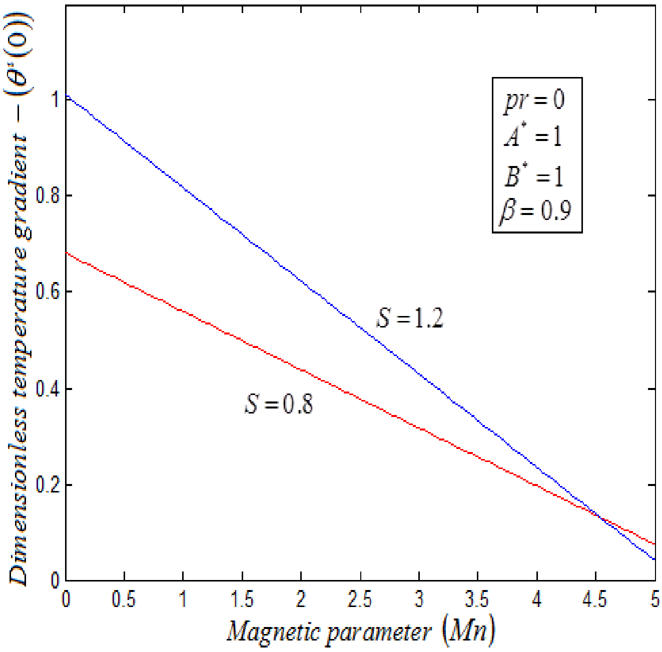

Fig.11: Dimensionless temperature gradient $-(\theta'(0))$ versus magnetic parameter (Mn). The curves are plotted for various values of the coefficient of space (S) and some fixed value of the other parameter pr,β,A^*,B^* using the eqn.(26) when $h=-0.65$.

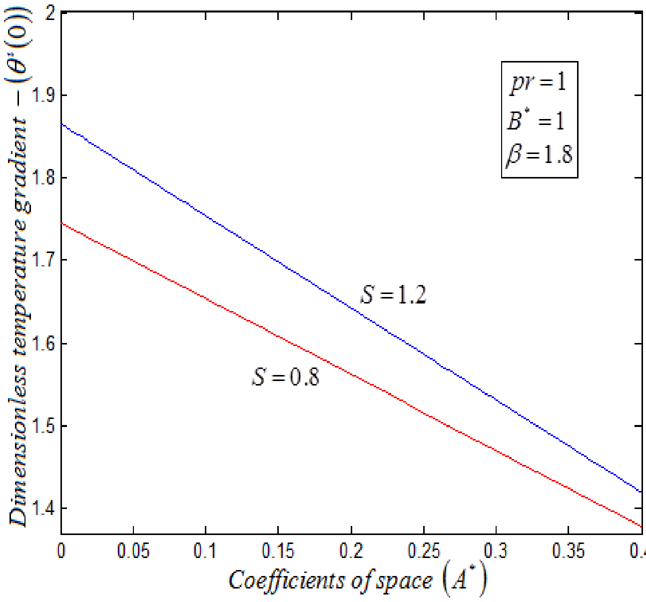

Fig.12: Dimensionless temperature gradient $-(\theta'(0))$ versus coefficient of space (A^*). The curves are plotted for various values of the coefficient of space (S) and some fixed value of the other parameter pr,β,B^* using the eqn.(26) when $h=0.252$.

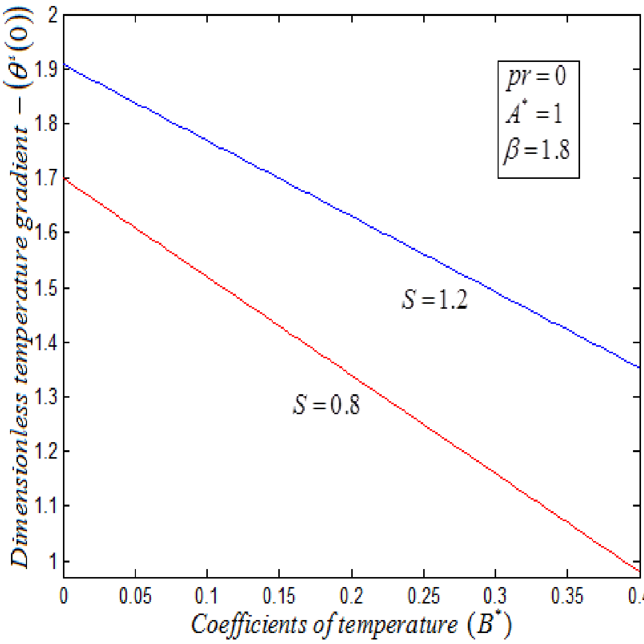

Fig.13: Dimensionless temperature gradient $-(\theta'(0))$ versus coefficient of temperature (B^*). The curves are plotted for various values of the coefficient of space (S) and some fixed

value of the other parameter pr, β, A^* using the eqn.(26) when $h = -0.885$.

V. CONCLUSION

This paper investigates the MHD boundary layer flow and heat transfer of a laminar liquid film over an unsteady stretching sheet. The analytical expressions of the dimensionless axial flow and dimensionless temperature are obtained by using New Homotopy analysis method. The effect of magnetic field and non-uniform source/sink in presented and magnetic field has a great effect in controlling the flow and heat transfer. In New Homotopy analysis method, we can choose h approximate way which controls the convergence of the series. This method can be easily extended to solve the non-linear boundary layer problems for MHD fluid flow in engineering field.

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APPENDIX A: BASIC CONCEPT OF HAM

Consider the following differential equation:

$$N[u(t)]=0 \quad \text{(A.1)}$$

Where N is a nonlinear operator, t denote an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao (2012) constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_0(t)] = phH(t)N[\varphi(t;p)] \quad \text{(A.2)}$$

where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is a nonzero auxiliary parameter, $H(t)\neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1$, it holds:

$$\varphi(t;0)=u_0(t) \text{ and } \varphi(t;1)=u(t) \quad \text{(A.3)}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t)p^m \quad \text{(A.4)}$$

Where

$$u_m(t)=\frac{1}{m!}\frac{\partial^m\varphi(t;p)}{\partial p^m}\Big|_{p=0} \quad \text{(A.5)}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (A.4) converges at $p=1$ then we have:

$$u(t)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t). \quad \text{(A.6)}$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and then setting $p=0$ and finally dividing them by $m!$, we will have the so-called mth -order deformation equation as:

$$L[u_m-\chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \quad \text{(A.7)}$$

where

$$\Re_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(t;p)]}{\partial p^{m-1}} \quad \text{(A.8)}$$

And

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \quad \text{(A.9)}$$

Applying L^{-1} on both side of eqn. (A7), we get

$$u_m(t)=\chi_m u_{m-1}(t)+hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \quad \text{(A10)}$$

In this way, it is easily to obtain u_m for $m\ge 1$, at M^{th} order, we have

$$u(t)=\sum_{m=0}^{M}u_m(t) \quad \text{(A.11)}$$

When $M\to+\infty$, we get an accurate approximation of the original eqn.(A.1). For the convergence of the above method we refer the reader to Liao [20]. If an eqn.(A.1) admits unique solution, then this method will produce the unique solution.

APPENDIX B

APPROXIMATE ANALYTICAL EXPRESSIONS OF THE NON-LINEAR DIFFERENTIAL EQNS. (16)-(20) USING THE NEW HOMOTOPY ANALYSIS METHOD

In this appendix,the eqns. (16) and (17) can be written in the following form

$$S(f'+\frac{\eta}{2}f'')+(f')^2-ff'=f'''-Mnf' \quad \text{(B.1)}$$

$$pr\left[\frac{S}{2}(3\theta+\eta\theta')+2\theta f'-\theta' f\right] = \theta''+(A^*f'+B^*\theta) \quad \text{(B.2)}$$

The eqns. (B.1) and (B .2) can be written as

$$f'''-Mnf'-S(f'+\frac{\eta}{2}f'')-(f')^2+ff'=0 \quad \text{(B.3)}$$

$$\theta''+(A^*f'+B^*\theta)-pr\left[\frac{S}{2}(3\theta+\eta\theta')+2\theta f'-\theta' f\right]=0 \quad \text{(B.4)}$$

We construct the Homotopy for the eqn. (B.3) and (B.4) are as follows:

$$(1-p)\left[\frac{d^3 f}{d\eta^3}\right]-hp\left[\begin{array}{l}\frac{d^3 f}{d\eta^3}-Mn\frac{df}{d\eta}\\ -S\left(\frac{df}{d\eta}+\frac{\eta}{2}\frac{d^2 f}{d\eta^2}\right)\\ -\left(\frac{df}{d\eta}\right)^2+f\frac{df}{d\eta}(0)\end{array}\right]=0 \quad (B.5)$$

$$(1-p)\left[\frac{d^2\theta}{d\eta^2}+pr\frac{d\theta}{d\eta}f\right]$$
$$-hp\left[\begin{array}{l}\frac{d^2\theta}{d\eta^2}+\left(A^*\frac{df}{d\eta}+B^*\theta\right)\\ -pr\left[\frac{S}{2}\left(3\theta+\eta\frac{d\theta}{d\eta}\right)\right]+2\theta\frac{df}{d\eta}-\frac{d\theta}{d\eta}f\end{array}\right]=0 \quad (B.6)$$

The approximate solution of the eqn. (B.5) and (B.6) are as follows:

$$f=f_0+pf_1+p^2 f_2+\dots \quad (B.7)$$

$$\theta=\theta_0+p\theta_1+\theta^2 T_2+\dots \quad (B.8)$$

The initial approximations are as follows:

$$f_0{}'(0)=1,\quad f_0(0)=0,\quad \theta_0(0)=1 \quad (B.9)$$

$$f_0{}''(\beta)=0\quad \theta_0{}'(\beta)=0 \quad (B.9\ (a))$$

$$f_i{}'(0)=1,\ f_i(0)=0,\ \theta_i(0)=1,\quad i=1,2,3,\dots \quad (B.10)$$

$$f_i{}''(\beta)=0;\theta_i{}'(\beta)=0,;\ i=1,2,3,\dots \quad (B.10\ (a))$$

$$f(\beta)=\frac{S\beta}{2}. \quad (B.11)$$

Substituting the eqn. (B.7) into the eqn. (B.5) we get

$$(1-p)\left[\frac{d^3(f_0+pf_1+\dots)}{d\eta^3}\right]$$
$$-hp\left[\begin{array}{l}\frac{d^3(f_0+pf_1+\dots)}{d\eta^3}-Mn\frac{d(f_0+pf_1+\dots)}{d\eta}\\ -S\left(\frac{d(f_0+pf_1+\dots)}{d\eta}+\frac{\eta}{2}\frac{d^2(f_0+pf_1+\dots)}{d\eta^2}\right)\\ -\left(\frac{d(f_0+pf_1+\dots)}{d\eta}\right)^2+f\frac{d(f_0+pf_1+\dots)}{d\eta}(0)\end{array}\right]=0 \quad (B.12)$$

Comparing the coefficients of like powers of p in eqn. (B.12), we get the following eqn.

$$p^0:\frac{d^3 f_0}{d\eta^3} \quad (B.13)$$

$$p^1:\frac{d^3 f_1}{d\eta^3}-h\left[\begin{array}{l}\frac{d^3 f_0}{d\eta^3}-Mn\frac{df_0}{d\eta}\\ -S\left(\frac{df_0}{d\eta}+\frac{\eta}{2}\frac{d^2 f_0}{d\eta^2}\right)-\left(\frac{df_0}{d\eta}\right)^2\\ +f\frac{df_0}{d\eta}(0)\end{array}\right] \quad (B.14)$$

Substituting the eqn. (B.8) into the eqn. (B.6) we get

$$(1-p)\left[\begin{array}{l}\frac{d^2(\theta_0+p\theta_1+\dots)}{d\eta^2}\\ +pr\frac{d(\theta_0+p\theta_1+\dots)}{d\eta}(f_0+pf_1+\dots)\end{array}\right]$$
$$-hp\left[\begin{array}{l}\frac{d^2(\theta_0+p\theta_1+\dots)}{d\eta^2}+\left(\begin{array}{l}A^*\frac{d(f_0+pf_1+\dots)}{d\eta}\\ +B^*\theta\end{array}\right)\\ -pr\left[\frac{S}{2}\left(3\theta+\eta\frac{d(\theta_0+p\theta_1+\dots)}{d\eta}\right)\right]\\ +2\theta\frac{d(f_0+pf_1+\dots)}{d\eta}-\frac{d(\theta_0+p\theta_1+\dots)}{d\eta}f\end{array}\right]=0 \quad (B.15)$$

Comparing the coefficients of like powers of p in eqn. (B.15), we get the following eqn.

$$p^0:\frac{d^2\theta_0}{d\eta^2}+pr\frac{d\theta_0}{d\eta}f \quad (B.16)$$

$$p^1:\left[\frac{d^2\theta_1}{d\eta^2}+pr\frac{d\theta_1}{d\eta}f\right]$$
$$-h\left[\begin{array}{l}\frac{d^2\theta_0}{d\eta^2}+\left(A^*\frac{df}{d\eta}+B^*\theta_0\right)\\ -pr\left[\frac{S}{2}\left(3\theta_0+\eta\frac{d\theta_0}{d\eta}\right)\right]+2\theta_0\frac{df}{d\eta}-\frac{d\theta_0}{d\eta}f\end{array}\right]=0 \quad (B.17)$$

Solving the eqns. (B.13), (B.14) and using boundary conditions (B.9) ,(B.9(a)) and (B.11), we obtained the following results:

$$f_0(\eta)=\eta+\frac{S\beta}{2}-\beta \quad (B.18)$$

$$f_1(\eta)=\left(\frac{Mn+S+1}{6}\right)\eta^3-\left(\frac{Mn+S+1}{2}\right)\beta\eta^2+\left(\frac{Mn+S+1}{3}\right)\beta^3 \quad (B.19)$$

Solving the eqns. (B.16), (B.17) and using boundary conditions (B.9) and (B.9(a)) we obtained the following results:

$$\theta_0(\eta)=1 \quad (B.20)$$

$$\theta_1(\eta)=\left[\begin{array}{l}\left(\dfrac{-A^*-B^*+\dfrac{3prS}{2}+2pr}{2}\right)\eta^2\\ -\left(\dfrac{A^*(Mn+S+1)}{24}\right)\eta^4\\ +\left(\dfrac{A^*(Mn+S+1)\beta}{6}\right)\eta^3\\ +\left(\dfrac{2pr(Mn+S+1)}{24}\right)\eta^4\\ -\left(\dfrac{2pr(Mn+S+1)\beta}{6}\right)\eta^3\\ +\left(A^*+B^*-\dfrac{3prS}{2}-2pr\right)\beta\eta\\ -\dfrac{A^*(Mn+S+1)\ \eta\ \beta^3}{3}\\ +\left(\dfrac{2pr(Mn+S+1)\beta^3}{3}\right)\eta\end{array}\right] \quad (B.21)$$

According to the Homotopy analysis method we have

$$f=\lim_{p\to 1} f(\eta)=f_0+f_1 \quad (B.22)$$

$$\theta=\lim_{p\to 1}\theta(\eta)=\theta_0+\theta_1 \quad (B.23)$$

Using the eqns. (B.18) - (B.21) in (B.18) – (B.19) respectively. We obtain the solution in the text eqns. (25)-(26).

APPENDIX C: NOMENCLATURE

Symbol	Meaning
η	Dimensionless similarity variable
f	Dimensionless velocity
f'	Dimensionless axial flow
θ	Dimensionless temperature
S	Unsteadiness parameter
Mn	Magnetic parameter
A^*	Coefficient of space
B^*	Coefficient of temperature
β	Dimensionless film thickness
pr	Prandtl number