*International Journal of Engineering Applied Sciences and Technology, 2021*
**Vol. 6, Issue 7, ISSN No. 2455-2143, Pages 16-22**
Published Online November 2021 in IJEAST (http://www.ijeast.com)

# ON THE PRIMITIVITY AND SOLUBILITY OF DIHEDRAL GROUPS OF DEGREE 3P THAT ARE NOT P-GROUPS BY NUMERICAL APPROACH

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***Abstract* — Let *G* be a dihedral group of degree 3p, p an odd prime. We apply some Group concepts and discuss primitivity and solubility by Numerical Approach. The method adopted uses a standard programme - Groups, Algorithms and Programming (GAP) to facilitate and validate our results.**

***Keywords*— Permutation Groups, Transitvity, Primitivity, Solubility, p-Groups and p-Subgroup.**

## I. INTRODUCTION

In mathematics, dihedral groups denoted by $D_n$ are the groups of symmetries of a regular polygon, which consist of rotations and reflections (Cameron, 2013). Dihedral groups are good examples of finite permutation groups and have series of applications especially in sciences and engineering.

Conventionally, we write

$$D_n = \langle r, f \mid r^n = f^2 = 1,\ fr = r^{n-1}f\ =\ r^{-1}f \rangle$$

And we say that $D_n$ is the group generated by the elements $r$ *and* $f$ subject to the conditions

$$r^n = f^2 = 1;\ fr = r^{n-1}f\ =\ r^{-1}f,$$

and the 2n distinct elements of $D_n$ are

$$1, r, r^2, \dots, r^{n-1}, f, rf, r^2f, \dots, r^{n-1}f.$$

Here $r$ is a rotation about the centre of the polygon through angle $2\pi/n$ and $f$ is a reflection about an axis of symmetry of the polygon.

When a group $G$ acts on a set $\Omega$, a typical point $\alpha$ is moved by elements of $G$ to various other points. The set of these images is called the orbit of $\alpha$ under $G$, and we denote it by $\alpha^G := \{\ \alpha^g \mid g \in G\ \}$. A group $G$ acting on a set $\Omega$ is said to be transitive on $\Omega$ if it has one orbit and so $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, $G$ is transitive if for every pair of point $\alpha$, $\beta\ \in\ \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$. A group which is not transitive is called intransitive.

A permutation group $G$ acting on a non empty set $\Omega$ is called primitive if $G$ acts transitively on $\Omega$ and $G$ preserves no non trivial partition of $\Omega$. Where non-trivial partition means a partition that is not a partition into singleton set or partition into one set $\Omega$. In other words, a group $G$ is said to be primitive on a set $\Omega$ if the only sets of imprimitivity are the trivial ones otherwise $G$ is imprimitive on $\Omega$. Transitive and Primitive finite permutation groups can be thought of as the building blocks of finite permutation groups, and questions about finite permutation groups can often be reduced to the primitive case (Fawcett, 2009).

According to (Cameron, 2013) a group is said to soluble/solvable if it has a normal series

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\} \dots\dots\dots (1)$$

such that each of its factor group

$$\frac{G_i}{G_{i+1}}, \qquad 0 \leq i \leq n$$

is an abelian group.

The above series (1) then is referred to as a solvable series of *G*.

Transitive, primitive and soluble permutation groups of special degrees have received much attention in the academic research space. Transitive and primitive p-subgroups of dihedral groups of degree p$q$, where p, $q$ are any two distinct odd prime numbers were considered by Hamma and Haruna (2009), while more recently, Audu and Hamma (2010) discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most $p^2$ using the concept of p-groups. They used the standard program – The Groups, Algorithms and Programming (GAP) to validate their results while Hamma and Aliyu, (2010) worked "On transitive and primitive dihedral groups of degree $at\, most\ 2^r\ (r \geq 2)$". Also, Hamma and Mohammed (2012) discussed the transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most $p^3$. They proved theorems and validate them using the Groups, Algorithms and Programming (GAP). Cai and Zhang, (2015) presented "A Note on Primitive Permutation Groups of Prime Power Degree". Fengler, (2018) in his published work explored on "Transitive Permutation Groups of Prime Degree" Studies concerning solubility include: Thanos (2006) who proved that $If\ |G| = p^k$ where p is a prime number then G is solvable. In other words every p-group where p is a prime number is solvable; Bello *et al.* (2017) used the concept of p-groups to construct locally solvable groups using two permutation groups by Wreath products. Gandi and Hamma, (2019) who investigated solvable and Nilpotent concepts on Dihedral Groups of an even degree regular polygon.

International Journal of Engineering Applied Sciences and Technology, 2021
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Published Online November 2021 in IJEAST (http://www.ijeast.com)

In this paper, we intend to obtain more detailed descriptions of the unique structure of dihedral groups of degree 3p that are not p-groups and discuss their primitivity and solubility using numerical approach.
In Section 2 we give some basic definitions, concepts and results which are required here. The main result of this paper covering all the dihedral groups of degree 3p where p is an odd prime number are stated in Section 3.

## II. PRELIMINARIES

The following preliminary definitions and results will be required.

### 2.1 p-Group
A finite group $G$ is said to be a $p$-group if its order is a power of $p$, where $p$ is prime.

### 2.2 p-Subgroups
Let $G$ be a group. Let H be a subgroup of $G$. if H is a $p$-group, then H is a p-subgroup of $G$.

### 2.3 Sylow Theorems (Sylow, 1872)
Let G be a finite group of order n.

1. If p is a prime such that $p^k$ is a divisor of |G| for some $k \geq 0$, then G contains a subgroup of order $p^k$.
2. All Sylow p-subgroups of G are conjugate, and any p-subgroup of G is contained in a Sylow p-subgroup.
3. Let $n = mp^k$, with (m, p) = 1, and let $n_p$ be the number of Sylow p-subgroups of G. Then $n_p \mid m$ and $n_p \equiv 1 \pmod p$.

### 2.4 Sylow p-Subgroup (Sylow, 1872)
Let $G$ be a group.

1. If $T \leq G$ and $|T| = p^r$, for some $r \geq 0$, then $T$ is called a $p$-subgroup of $G$.
2. If $G$ is finite and $|G| = p^r m$, $r \geq 1$ where $p$ and $m$ are co-prime and $H \leq G$ such that $|H| = p^r$, we say that $H$ is a Sylow $p$-subgroup of $G$.

Clearly, a Sylow $p$-subgroup is maximal among all $p$-subgroups of $G$.
According to Sylow theorem, if $n$ divides $|G|$, then $G$ has a subgroup of order $n$ provided that $n$ is a prime power.

This result is a sufficient condition for a subgroup to exist and is one of the basic tools in modern finite group theory.

### 2.5 Transitivity
A group $G$ is transitive if for every pair of point $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g \in \beta$. A group which is not transitive is called intransitive.
If $|\Omega| \geq 2$, we say that the action of $G$ on $\Omega$ is doubly transitive iff for any $\alpha_1, \alpha_2 \in \Omega$ such that $\alpha_1 \neq \alpha_2$ and $\beta_1, \beta_2 \in \Omega$ such that $\beta_1, \neq \beta_2$ there exist $g \in G$ such that $\alpha_1^g = \beta_1$, $\alpha_2^g = \beta_2$.
The group $G$ is said to be k-transitive (or k-fold transitive) on $\Omega$ if for any sequences $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_i \neq \alpha_j$ when $i \neq j$ and $\beta_1, \beta_2, \ldots, \beta_k$ such that $\beta_i \neq \beta_j$ when $i \neq j$ of $k$ element on $\Omega$, there exists $g \in G$ such that $\alpha_i^g = \beta_i$ for $1 \leq i \leq k$.
Examples:
(i) $A_4$ ={(1), (2 4 3), (2 3 4), (1 4 3), (1 4)(2 3), (1 4 2), (1 3 4), (1 3 2), (1 3)(2 4), (1 2 4), (1 2)(3 4), (1 2 3)} is transitive.

(ii) K = {(1), (12), (34), (12) (34)} is intransitive.

(iii) The Group $D_6$ is doubly transitive as, if $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = 1$ and $\beta_2 = 3$ then routine calculation shows that there exist $g \in D_6$ such that $\alpha_1^g = \beta_1$ and $\alpha_2^g = \beta_2$

### 2.6 Lemma (Passman, 1968)
Let G be a dihedral group of any order, then G is transitive.
**Proof**
For given $\alpha_i$, $\alpha_j$ as any two vertices of the regular polygon with i < j, we readily see that $(\alpha_1\alpha_2 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi c/n$, (where n is the number of edges of the polygon) which takes α to αj . As such G is transitive

### 2.7 Some Results On Transitive Groups
Let $G$ be a permutation group on Ω, where Ω is a finite set.

1. We say that $G$ is $\frac{1}{2}$ – transitive if all the orbits have the same size.
2. Suppose that $G$ has just one orbit Ω. then for all $r \in \Omega$, $r^G =$ Ω and as such for any $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$, and $G$ is said to be transitive (or that $G$ acts transitively) on Ω
3. The group G is said to be k-fold transitive (or, simply k-transitive) on Ω if, for any sequences $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_i \neq \alpha_j$ when $i \neq j$; $\beta_1, \beta_2, \ldots, \beta_k$ such that $\beta_1 \neq \beta_j$ when $i \neq j$ of $k$ elements of Ω, there exists $g \in$ G such that
$$\alpha_i^g = \beta_i \text{ for } 1 \leq i \leq k.$$
Thus for $k = 2$ we have that for $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Ω with $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ there exists $g \in G$ such that;
$$\alpha_1^g = \beta_1,\ \alpha_2^g = \beta_2$$
and we say that G is doubly transitive.

If $k \geq 2$ then $k-$ transitivity implies $(k-1)$ $-$ transitively.
4. Let $G$ act on itself by right multiplication. Then, $\Omega = G$. If $\alpha = x$, $\beta = y$ in Ω and we take $g = x^{-1}y$; then $\alpha^g = x(x^{-1}y) = y = \beta$.

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and so $G$ is transitive.
Let $H \leq G$ and let $G$ act on right cosets of $H$ in $G$.
Then $G$ is transitive on $\Omega := (G : H)$. For if $\alpha, \beta \in \Omega$, then $\alpha = H, \beta = H_y$ for some $x, y \in G$, and if we take $g := x^{-1}y$ then we have

$$\alpha^g = (Hx)x^{-1}y = Hy = \beta$$

### 2.8 Primitivity

A permutation group $G$ is said to be primitive on a set $\Omega$ if the only sets of imprimitivity are the trivial ones otherwise $G$ is imprimitive on $\Omega$. For example the group,
$S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is primitive as $\{1,2\}^{(1\,2\,3)} = \{2,3\}$ implying that $\Delta^g \neq \Delta$ and $\Delta^g \cap \Delta \neq \emptyset$ for $\Delta = \{1,2\} \in \Omega$.. $\Delta$ of $\Omega$ is said to be a set of imprimitivity for the action of $G$ on $\Omega$, if for each $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g$ and $\Delta$ are disjoint. In particular, $\Omega$ itself, the 1-element subsets of $\Omega$ and the empty set are obviously sets of imprimitivity which are called trivial set of imprimitivity.
The group of symmetry $D_4 = \{(1), (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\}$ of the square with vertices 1,2,3,4 is imprimitive. For take $G_1 = \{(1), (24)\}$.
Let $H = \{(1), (13), (24), (13)(24)\}$ which is a normal subgroup of $G$. Then H is a group greater than $G_1$, but not equal to $G$.

### 2.9 Theorem

Let G be a transitive permutation group of prime degree on $\Omega$. Then G is primitive
**Proof**
Now since G is transitive, it permutes the sets of imprimitivity bodily and all the sets have the same size. But $\Omega = \cup |\Omega_i|$, $\Omega_i$ being the sets of imprimitivity. As $|\Omega|$ is prime we have that either each $|\Omega_i| = 1$ $or$ $\Omega$ or the only sets of imprimitivity. So, G is primitive.

### 2.10 Theorem (Passman, 1968).

Let G be a non-trivial transitive permutation group on $\Omega$. Then G is primitive if and only if $G_\alpha, \alpha \in \Omega$ is a maximal subgroup of G or equivalently G is imprimitive if and only if there is a subgroup H of G properly lying between $G_\alpha (\alpha \in \Omega)$ and G.
**Proof**
Suppose G is imprimitive and $\Psi$ a non-trivial subset of imprimitivity of G.
Let $H = \langle g \in G | \Psi^g = \Psi \rangle$
Clearly H is a subgroup of G and a proper subgroup of G because $\Psi \subset \Omega$ and G is transitive.
Now choose $\alpha \in \Psi$. If $g \in G$ then $\alpha \in \Psi \cap \Psi^g$ and so $\Psi = \Psi^g$.
Hence $G_\alpha \leq H \leq G$
Since $|\Psi| \neq 1$, choose $\beta \in \Psi$ such that $\beta \neq \alpha$. By transitivity of G, there exists some $h \in G$ with $\alpha^h = \beta$ so that $h \in G_\alpha$. Now $\beta \in \Psi \cap \Psi^h$, so $\Psi = \Psi^h$ and $h \in H - G_\alpha$. Thus $H \neq G_\alpha$
Conversely, suppose that $G_\alpha < H < G$ for some subgroup H.
Let $\Psi = \alpha^H$. Since $H > G_\alpha |\Psi| \neq 1$
Now if $\Psi = \Omega$, then H is transitive on $\Omega$ and hence $|\Omega| = |H: G_\alpha|$ showing that H = G, a contradiction.
Hence, $\Psi \neq \Omega$
Now we shall show that $\Psi$ is a subset of imprimitivity of G.
Let $g \in G$ and $\beta \in \Psi \cap \Psi^g$ then $\beta = \alpha^h = \alpha^{h\,g}$ for some $h, h \in H$.
Hence $\alpha^{h\,gh^{-1}} = \alpha$. so $h\,gh^{-1} \in G_\alpha < H$
This shows that $g \in H$
Thus $\Psi = \Psi^g$. Hence $\Psi$ is a non-trivial subset of imprimitivity
So G is imprimitive.

### 2.11 Theorem

A group G is solvable if and only if it has a solvable series.
**Proof**
Suppose $G$ is solvable. Then by the definition of "solvable," in the derived series of commutator subgroups we have $G^{(n)} = (1)$, for some n $\in N$. In the series $G > G^{(1)} > G^{(2)} > \ldots > G^{(n)} = (1)$, we have that $G^{(i+1)}$ is normal in $G^{(i)}$ and $G^{(i)}/G^{(i+1)}$) is abelian. So the series is subnormal (because each subgroup is normal in each previous subgroup) and is also solvable (since the quotient groups are abelian).
Now suppose $G = G_0 > G_1 > \ldots > G_n = (l)$ is s solvable series. Then $G_i/G_{i+1}$ is abelian (by definition of solvable series) for $0 \leq i \leq n - l$. $G_{i+l} > (G_i)'$ for $0 \leq i \leq n - 1$. Since in the derived series of commutator subgroups we have $G > G^{(1)} > G^{(2)} > \ldots > G^{(n)}$, then
$G_1 > G_0' = G' = G^{(1)}$
$G_2 > G_1' = (G^{(1)})' = G^{(2)}$
$G_3 > G_2' = (G^{(2)})' = G^{(3)}$
$G_{i+1} > G'_i = (G^{(i)})' = G^{(i+1)}$
$G_n > G'_{n+1} = (G^{(n-1)})' = G^{(n)}$
But $G_n = (1)$ so it must be that $G^{(n)} = (1)$ and G is solvable.

### 2.12 Corollary

Let $G$ be a finite group and $H$ a Sylow p-subgroup of $G$. Then $H$ is the only Sylow p-subgroup of $G$ if and only if $H$ is normal in $G$.
**Proof:**
By Sylow theorem, the Sylow p-subgroups of $G$ are the elements of the sets $\{g^{-1}Hg \mid g \in G\}$ and this reduces to a singleton set if and only if $g^{-1}Hg = H$ for all $g \in G$; that is precisely when $H$ is normal in $G$.

### 2.13 Corollary (Thonas, 2006)

If $H \trianglelefteq G$ and $\left|\frac{G}{H}\right| = p$ or $p^2$ then $\frac{G}{H}$ is abelian

### 2.14 Proposition (Thonas, 2006)

Let G be solvable and $H \leq G$. Then
1. $H$ is solvable.
2. If $H \lhd G$, then $G/H$ is solvable.
**Proof**

Start from a series with abelian slices. G: $G_0 \rhd G_l \rhd \ldots \rhd G_n = (l)$ Then $H = H \cap G_0 \rhd H \cap G_1 \rhd \ldots \rhd H \cap G_n = \{1\}$. When $H$ is normal, we use the canonical projection $\pi$: G → G/ H to get G/ H = $\pi(G_0) \rhd \ldots \pi(G_n) = \{1\}$ ; the quotients are abelian as well, so G / H is still solvable.

## III. RESULTS

### 3.1 Theorem

Let $\Omega = \{1,2,3, \ldots\ldots, 3p\}$ where p is an odd prime number. The following occurs:

### 3.1.1 The dihedral group $D_{3p}$ of degree 3p, p = 3 is (i) imprimitive and (ii) soluble.

**Proof:**

Now $D_9$ is the dihedral group of order $2 \times n = 18$ and $\Omega = \{1,2,3,4,5,6,7,8,9\}$ is the set of points of $D_9$. We readily see that $D_9$ is transitive as the orbit $\alpha^{D_9} = \Omega\ \forall\, \alpha \in \Omega$.

The elements of the dihedral group of degree 9 are:

$D_9$ = {(1), (2,9)(3,8)(4,7)(5,6), (1,2)(3,9)(4,8)(5,7), (1,2, 3,4, 5,6,7,8,9), (1,3)(4,9)(5,8)(6,7), (1,3,5,7,9,2,4,6,8), (1,4)(2,3)(5,9)(6,8), (1,4,7)(2,5,8)(3,6,9), (1,5)(2,4)(6,9)(7,8), (1,5,9,4, 8,3,7,2,6), (1,6)(2,5)(3,4)(7,9), (1,6,2,7,3,8,4,9,5), (1,7) (2,6)(3,5)(8,9), (1,7,4)(2,8,5)(3,9,6), (1,8)(2,7)(3,6)(4,5), (1,8,6,4, 2,9,7,5,3), (1,9,8,7,6,5,4,3,2), (1,9)(2,8)(3,7)(4,6)}

The stabilizer of the point 1 in $D_9$ is given by $D_{9_{\{1\}}} = \{(1), (2,9)(3,8)(4,7)(5,6)\}$ which is obviously a non-identity proper subgroup of $D_9$. We readily see from the group elements that the group $D_9$ has a subgroup H = {(1), (1,4,7)(2,5,8)(3,6,9), (2,9)(3,8)(4,7)(5,6)} properly lying between $D_{9_{\{1\}}}$ and $D_9$ that is, $D_{9_{\{1\}}} < H < D_9$. Thus by virtue of Theorem 2.10, $G$ is imprimitive, proving (1).

(ii) Now, $|D_9| = 2 \times 9 = 18 = 2 \times 3^2$.

Let $H_2 = Syl_2(D_9)$ and $H_3 = Syl_3(D_9)$ be the Sylow 2-subgroups and Sylow 3-subgroups of $D_9$ respectively. Routine calculation shows that $D_9$ has:

$H_2$ = {(1), (2,9)(3,8)(4,7)(5,6)} ≤ $D_9$ with $|Syl_2(D_9)| = 2$ and
$H_3$ = {(1), (1,2,3,4,5,6,7,8,9),
(1,3,5,7,9,2,4,6,8),(1,4,7)(2,5,8)(3,6,9), (1,5,9,4, 8,3,7, 2,6),
(1,6,2,7,3,8,4,9,5), (1,7,4)(2,8,5)(3,9,6), (1,8,6,4,2,9,7,5,3),
(1,9,8,7,6,5,4,3,2)} ≤ $D_9$ with $|Syl_3(D_9)| = 9$

Going by theorem 2.3, the number of Sylow 2-subgroups of $D_9$ denoted $n_2$ is given by $n_2 = 1 + 2k \equiv 1 \pmod 2$ and $n_2 \mid 9$ (where k = {0, 1, 2, ….}). Therefore $n_2 = 1$ or 3 or 9 implying that $n_2$ is not unique and hence not normal in $D_9$.

Similarly the number of Sylow 3-subgroups of $D_9$ denoted $n_3$is given by $n_3 = 1 + 3k \equiv 1 \pmod 3$ and $n_3 \mid 2$ (where k = {0, 1, 2, …}).

It follows from the constraints that $n_3 = 1$ implying it is unique and its normal in $D_9$ therefore, $D_9$ has a normal series

$$D_9 \rhd H_3 \rhd (1)$$

with factor groups $D_9/H_3$ and $H_3/(1)$ = 2 and $3^2$ respectively which are either abelian or cyclic by theorems 2.17. $D_9/H_3$ and $H_3/(1)$ are solvable by theorems 2.14. It follow that $D_9$ is solvable by theorem 2.11.

### 3.1.2 The dihedral group $D_{3p}$ of degree 3p, p = 5 is (i) imprimitive and (ii) soluble.

**Proof:**

Now $D_{15}$ is the dihedral group of order $2 \times n = 30$ and $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ is the set of points of $D_{15}$. We readily see that $D_{15}$ is transitive as the orbit $\alpha^{D_9} = \Omega\ \forall\, \alpha \in \Omega$.

The elements of the dihedral group of degree 15 are:

$D_{15}$ = {(1), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9), (1,2) (3, 15) (4,14)(5,13) (6,12) (7,11) (8,10),(1,2,3,4,5,6, 7,8,9,10, 11,12,13,14,15),(1,3)(4,15)(5,14)(6,13)(7,12)(8,11)(9,10),(1,3, 5,7, 9,11,13,15,2,4,6,8,10,12,14),(1,4)(2,3)(5,15)(6,14)(7,13)(8,12)(9,11),(1,4,7,10,13)(2,5,8,11,14 )(3,6,9,12,15),(1,5) (2,4)(6,15)(7,14)(8,13)(9,12)(10,11),(1,5,9,13,2,6,10,14,3,7,11,15,4 ,8, 12), (1,6)(2,5)(3,4)(7,15)(8,14)(9,13)(10,12),(1,6,11) (2,7, 12) (3,8,13)(4,9,14)(5,10,15),(1,7) (2,6) (3,5) (8, 15)(9,14)(10,13)(11,12), (1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12), (1,8)(2,7)(3,6) (4,5) (9,15) (10, 14)(11,13), (1,8,15,7,14,6,13,5, 12,4,11,3,10,2,9),(1,9)(2,8)(3,7)(4,6)(10,15) (11,14) (12,13), (1,9, 2,10,3,11,4,12,5,13,6,14,7,15,8),(1,10)(2,9)(3,8)(4,7) (5, 6) (11,15)(12,14), (1,10, 4,13,7) (2,11, 5, 14,8)(3,12,6,15,9), (1,11)(2,10)(3,9)(4,8)(5,7)(12,15)(13,14), (1,11,6)(2,12,7) (3, 13,8) (4,14,9) (5,15,10),(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)(13, 15), (1,12,8,4,15,11,7,3,14,10, 6,2,13,9,5), (1,13) (2,12) (3,11) (4,10)(5,9)(6,8)(14,15), (1,13,10,7,4)(2,14,11,8,5)(3,15,12,9, 6), (1,14)(2,13) (3,12) (4,11)(5,10)(6,9)(7,8), (1,14,12,10,8, 6,4,2,15, 13,11,9,7,5,3), (1,15,14,13, 12,11, 10,9, 8, 7, 6,5,4, 3,2), (1,15)(2,14)(3,13)(4,12)(5,11)(6,10)(7,9)}

The stabilizer of the point 1 in $D_{15}$ is given by $D_{15_{\{1\}}} = \{(1), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)\}$ which is obviously a non-identity proper subgroup of $D_{15}$.

We readily see from the group elements that the group $D_{15}$ has a subgroup H = {(1),
(1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15),
(2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)} properly lying between $D_{15_{\{1\}}}$ and $D_{15}$ that is, $D_{15_{\{1\}}} < H < D_{15}$. Thus by virtue of Theorem 2.10, $D_{15}$ is imprimitive, proving (1).

(ii) Now, $|D_{15}| = 2 \times 15 = 18 = 2 \times 3 \times 5$.

Let $H_2 = Syl_2(D_{15})$, $H_3 = Syl_3(D_{15})$ and $H_5 = Syl_3(D_{15})$ be the Sylow 2-subgroups, Sylow 3-subgroups and Sylow 5-subgroups of $D_{15}$ respectively. Routine calculation shows that $D_{15}$ has:

$H_2$ = {(1), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)} ≤ $D_{15}$ with $|Syl_2(D_{15})| = 2$,
$H_3$ = {(1), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15), (1,11, 6) (2,12,7)(3,13,8)(4,14,9)(5,15,10)} ≤ $D_{15}$ with $|Syl_3(D_{15})| = 3$
and $H_5$ = {(1), (1,4,7,10,13)(2,5,8,11,14) (3, 6,9,12,15), (1,7, 13,4,10) (2, 8,14,5,11)(3,9,15,6,12),(1,10, 4, 13,7) (2,11, 5,14, 8)(3,12,6,15,9), (1,13,10,7,4)(2,14,11,8,5) (3, 15, 12,9,6)} ≤ $D_{15}$ with $|Syl_3(D_{15})| = 5$

Going by theorem 2.3, the number of Sylow 2-subgroups of $D_{15}$ denoted $n_2$ is given by $n_2 = 1 + 2k \equiv 1 \pmod 2$ and $n_2 \mid 15$ (where k = {0, 1, 2, ….}). Therefore $n_2 = 1$ or 3 or 5 or 15 implying that $H_2$ is not unique and hence not normal in $D_{15}$. Also the number of Sylow 3-subgroups of $D_{15}$ denoted $n_3$ is given by $n_3 = 1 + 3k \equiv 1 \pmod 3$ and $n_3 \mid 10$ (where k = {0, 1, 2, …}). Therefore $n_3 = 1$ or 10 implying that $H_3$ is not unique and hence not normal in $D_{15}$. Also the number of Sylow 5-subgroups of $D_{15}$ denoted $n_5$ is given by $n_5 = 1 + 5k \equiv 1 \pmod 5$ and $n_5 \mid 6$ (where k = {0, 1, 2, …}). $n_5 = 1$ or 6 thus $H_5$ is not unique and hence not normal in $D_{15}$
However the conjugacy classes of $D_{15}$ shows there are 4NO subgroups of order 15. Let $H_{15}$ be a subgroup of order 15 then, $D_{15}$ has a normal series

$$D_{15} \rhd H_{15} \rhd D_5 \rhd (1)$$

with factor groups $D_{15}/H_{15}$, $H_{15}/H_5$ and $H_5/(1) = 2$, 3 and 3 respectively which are either abelian or cyclic by theorems 2.13. $D_{15}/H_{15}$, $H_{15}/H_5$ and $H_5/(1)$ are solvable by theorems 2.14. It follow that $D_{15}$ is solvable by theorem 2.11.

The main results obtain from the investigation of dihedral groups are as follows:

### 3.2.1 Proposition (Main Result)

Let $G$ be a dihedral group of degree 3p, p an old prime number. Then $G$ is (i) imprimitive and (ii) soluble.

**Proof**

That G is transitive follows easily from Lemma 2.18. Next, name the vertices of $G$ as 1,2,3,…,3p and let $l$ be the line of symmetry joining the vertex 1 and the middle of the vertices

$\frac{3p+1}{2}$ *and* $\frac{3p+3}{2}$ *so that* $\alpha = (2, 3p)(3, 3p-1)(4, 3p-2) \ldots \ldots \left(\frac{3p+1}{2}, \frac{3p+3}{2}\right)$ is the reflection in $l$ (see figure 1). Then $G_{(1)} = \{(1), \alpha\}$ is the stabilizer of the point 1. We readily see that $G_{(1)}$ is a non-identity proper subgroup of $G$ which has

$H = \left\{(1), (2, 3p), (3, 3p-1), (4, 3p-2), \ldots, \left(\frac{3p+1}{2}, \frac{3p+3}{2}\right), \alpha\right\}$ as a subgroup properly lying between $G_{(1)}$ and $G$. It follows by virtue of Theorem 2.10 that $G$ is imprimitive, proving (i).

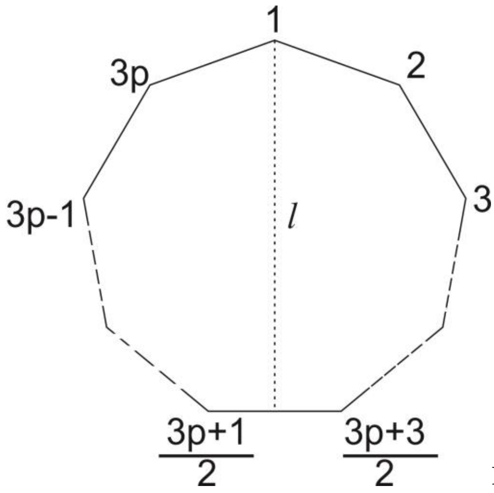


Figure 1

(ii) Now, the order $|G_{3p}| = 2(3p)$.
There are two cases here,
Case 1: p = 3 then, $|G_{3p}| = 2(3p) = 18 = 2 \cdot 3^2$
By corollary 2.3, the number of Sylow 3-subgroups $n_3 \equiv 1 (\text{mod} 3)$ and $n_3$ divides 2.
It follows from this constraints that $n_3(G_9) = 1$.
Hence the Sylow 3-subgroup $H$ is unique and is a normal subgroup of $G_9$.
The order of $H$ is 9, a square of a prime number, thus $H$ is abelian by Corollary 2.13.
Also, the order of the quotient group $G_9/H$ is 2, thus $G_9/H$ is an abelian (cyclic) group.
Thus we have the subnormal series

$$G_9 \rhd H \rhd \{e\}$$

whose factors $G_{3p}/H$, $H/\{e\}$ are abelian groups, hence $G_{3p}$ is solvable.

Case 2: p > 3 then, $|G_{3p}| = 2(3p) = 6p$
By corollary 2.10, the number of Sylow p-subgroups $n_p \equiv 1 (\text{mod p})$ and $n_p$ divides 6.
This implies $n_p = 1 + kp$ where k = {0, 1, 2, ….} ≡ 1(mod p) and $n_p \mid 6$.
It follows from this constraints that $n_p(G_{3p}) = 1$ or 6( only when p = 5).
Hence the Sylow p-subgroup $H$ is unique and is a normal subgroup of $G_{3p}$.
The order of $H$ is p, a prime number, thus $H$ is abelian by Corollary 2.13.
The order of the factor group $G_{3p}/H$ that is, $|G_{3p}/H| = 6$ and there are two groups of order 6 namely $Z_6$ and $S_3$ which are solvable groups. We also know that Sylow groups are generally solvable. Since $G_{3p}/H$ is solvable and $H$ is solvable, it follows by Theorem 2.11 that $G_{3p}$ is soluble.

### 3.2.2 GAP Result - Validation

GAP 4.11.1 of 2021-03-02
GAP https://www.gap-system.org

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Published Online November 2021 in IJEAST (http://www.ijeast.com)

```
└──────────┘  Architecture: x86_64-pc-cygwin-default64-
kv7
 Configuration:  gmp 6.2.0, GASMAN, readline
 Loading the library and packages ...
gap>
gap>
gap> D9 := DihedralGroup(IsGroup, 18);
Group([ (1,2,3,4,5,6,7,8,9), (2,9)(3,8)(4,7)(5,6) ])
gap> Order(D9);
18
gap> Elements(D9);
[ (), (2,9)(3,8)(4,7)(5,6), (1,2)(3,9)(4,8)(5,7),
(1,2,3,4,5,6,7,8,9), (1,3)(4,9)(5,8)(6,7),
  (1,3,5,7,9,2,4,6,8), (1,4)(2,3)(5,9)(6,8), (1,4,7)(2,5,8)(3,6,9),
(1,5)(2,4)(6,9)(7,8),
  (1,5,9,4,8,3,7,2,6), (1,6)(2,5)(3,4)(7,9), (1,6,2,7,3,8,4,9,5),
(1,7)(2,6)(3,5)(8,9),
  (1,7,4)(2,8,5)(3,9,6), (1,8)(2,7)(3,6)(4,5), (1,8,6,4,2,9,7,5,3),
(1,9,8,7,6,5,4,3,2),
  (1,9)(2,8)(3,7)(4,6) ]
gap> IsTransitive(D9);
true
gap> IsPrimitive(D9);
false
gap> IsSolvable(D9);
true
gap> S2 := SylowSubgroup(D9, 2);
Group([ (2,9)(3,8)(4,7)(5,6) ])
gap> Order(S2);
2
gap> Elements(S2);
[ (), (2,9)(3,8)(4,7)(5,6) ]
gap> S3 := SylowSubgroup(D9, 3);
Group([ (1,2,3,4,5,6,7,8,9), (1,4,7)(2,5,8)(3,6,9) ])
gap> Order(S3);
9
gap> Elements(S3);
[ (), (1,2,3,4,5,6,7,8,9), (1,3,5,7,9,2,4,6,8),
(1,4,7)(2,5,8)(3,6,9), (1,5,9,4,8,3,7,2,6),
  (1,6,2,7,3,8,4,9,5), (1,7,4)(2,8,5)(3,9,6), (1,8,6,4,2,9,7,5,3),
(1,9,8,7,6,5,4,3,2) ]
gap>
gap>
gap> D15 := DihedralGroup(IsGroup, 30);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15),
(2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9) ])
gap> Order(D15);
30
gap> Elements(D15);
[ (), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9),
(1,2)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10),
  (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15),
(1,3)(4,15)(5,14)(6,13)(7,12)(8,11)(9,10), (1,3,5,7,
   14), (1,4)(2,3)(5,15)(6,14)(7,13)(8,12)(9,11),
(1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15),
  (1,5)(2,4)(6,15)(7,14)(8,13)(9,12)(10,11),
(1,5,9,13,2,6,10,14,3,7,11,15,4,8,12),
  (1,6)(2,5)(3,4)(7,15)(8,14)(9,13)(10,12),
(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15),
  (1,7)(2,6)(3,5)(8,15)(9,14)(10,13)(11,12),
(1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12),
  (1,8)(2,7)(3,6)(4,5)(9,15)(10,14)(11,13),
(1,8,15,7,14,6,13,5,12,4,11,3,10,2,9),
  (1,9)(2,8)(3,7)(4,6)(10,15)(11,14)(12,13),
(1,9,2,10,3,11,4,12,5,13,6,14,7,15,8),
  (1,10)(2,9)(3,8)(4,7)(5,6)(11,15)(12,14),
(1,10,4,13,7)(2,11,5,14,8)(3,12,6,15,9),
  (1,11)(2,10)(3,9)(4,8)(5,7)(12,15)(13,14),
(1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10),
  (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)(13,15),
(1,12,8,4,15,11,7,3,14,10,6,2,13,9,5), (1,13)(2,1
   8)(14,15), (1,13,10,7,4)(2,14,11,8,5)(3,15,12,9,6),
(1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7
  (1,14,12,10,8,6,4,2,15,13,11,9,7,5,3),
(1,15,14,13,12,11,10,9,8,7,6,5,4,3,2), (1,15)(2,14)(
   10)(7,9) ]
gap> IsTransitive(D15);
true
gap> IsPrimitive(D15);
false
gap> IsSolvable(D15);
true
gap> S2 := SylowSubgroup(D15, 2);
Group([ (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9) ])
gap> Order(S2);
2
gap> Elements(S2);
[ (), (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9) ]
gap> S3 := SylowSubgroup(D15, 3);
Group([ (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10) ])
gap> Order(S3);
3
gap> Elements(S3);
[ (), (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15),
(1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10) ]
gap> S5 := SylowSubgroup(D15, 5);
Group([ (1,10,4,13,7)(2,11,5,14,8)(3,12,6,15,9) ])
gap> Order(S5);
5
gap> Elements(S5);
[ (), (1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15),
(1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12),
  (1,10,4,13,7)(2,11,5,14,8)(3,12,6,15,9),
(1,13,10,7,4)(2,14,11,8,5)(3,15,12,9,6) ]
gap> CCD15 := ConjugacyClasses(D15);
[ ()^G, (2,15)(3,14)(4,13)(5,12)(6,11)(7,10)(8,9)^G,
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,
   15)^G, (1,3,5,7,9,11,13,15,2,4,6,8,10,12,14)^G,
(1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,
```

*International Journal of Engineering Applied Sciences and Technology, 2021*
**Vol. 6, Issue 7, ISSN No. 2455-2143, Pages 16-22**
Published Online November 2021 in IJEAST (http://www.ijeast.com)

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   15)^G, (1,5,9,13,2,6,10,14,3,7,11,15,4,8,12)^G,
(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,
   15)^G, (1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12)^G,
(1,8,15,7,14,6,13,5,12,4,11,3,10,2,
   9)^G ]
gap> List(CCD15, x -> Order(Representative(x)));
[ 1, 2, 15, 15, 5, 15, 3, 5, 15 ]
gap>
```

## IV. Conclusion and Recommendation

### 4.1 Conclusion

This study showed that Dihedral group of degree 3p where p is an odd prime number is (i) imprimitive and (ii) soluble.

### 4.2 Recommendation

This study can be extended by considering for further research, one or a combination of two or more of other theoretic properties such as simplicity, nilpotency, regularity, etc of same algebraic structures.

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